DCU-003-1161002

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Seat No. ________

M. Sc. (Sem. I) (CBCS) Examination

August – 2022

Mathematics : Paper - CMT-1002

(Real Analysis)

Faculty Code : 003
Subject Code : 1161002

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Answer any five questions.

(2) Each questions carries 14 marks.

(3) There are 10 questions in total.

1 Answer the following **seven** questions : **14**

(1) Define : σ-algebra of a non-empty set X.

(2) Let R be an algebra of sets on non empty set X. Let $A\Delta B = (A-B)\cup(B-A)$. Then for $A, B \in R$ show that, $A\Delta B \in R$.

(3) Let $X \neq \phi$ and R be σ-algebra on X. Let $A_1, A_2, \ldots, A_n, \ldots \in R$. Then prove that $\bigcap_{i=1}^{\infty} A_i \in R$.

(4) Define : $F_\sigma -$ sets. Justify that, an open interval is a $F_\sigma -$ set.

(5) Using outer measure, prove that [1, 2) is not a countable subset of $\mathbb{R}$.

(6) Let $A, B \subseteq \mathbb{R}$ and $m^*B = 0$. Then prove that, $m^*(A\cup B) = m^*A$.

(7) Define : Measurable function. Also given an example of a measurable function on $\mathbb{R}$.

2 Answer the following seven questions : **14**

(1) Let $A, B \subseteq \mathbb{R}$ and $A \subseteq B$. Then prove that, $m^*A \leq m^*B$.

(2) Write down $m^*(\mathbb{N} \times \mathbb{N})$ and $m^*([1, 3] \cap \mathbb{Q})$.

(3) Define: Lebesgue measurable set.

(4) Define: Characteristic function and Simple function.

(5) Write down all three Littlewood's principles without proof.

(6) Is Cantor set a measurable set? Justify your answer.

(7) Define: Almost everywhere property.

3 Answer the following two questions : **14**

(1) Let $X \neq \phi$ and $R \subseteq P(X)$. Suppose R satisfies the condition that if $A \in R$. Then prove that, the following statements are equivalent :

(i) R is an algebra of sets on X.

(ii) $A_1, A_2, \ldots, A_n \in R$ then $\bigcup_{i=1}^{n} A_i \in R$.

(iii) $A_1, A_2, \ldots, A_n \in R$ then $\bigcap_{i=1}^{n} A_i \in R$.

(iv) $A, B \in R$ then $A \cap B \in R$.

(2) Let $X \neq \phi$ and C be a collection of subsets of X. Let $\beta = \{R / R \text{ is a boolean algebra on } X \text{ and } C \subseteq R\}$ and $H = \bigcap_{R \in \beta} R$. Then prove that, H is the smallest Boolean algebra on X, which contains C.

4 Answer the following two questions : **14**

(1) Let β_1 be the σ-algebra on $\mathbb{R}$, generated by the collection of all closed sets in $\mathbb{R}$ and β_2 be the σ-algebra on $\mathbb{R}$, generated by the collection of all open sets in $\mathbb{R}$. Then prove that, $\beta_1 = \beta_2 = B_0$, the Borel field on $\mathbb{R}$.

(2) Let $\langle A_n \rangle \subseteq P(\mathbb{R})$. Then prove that, $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^* A_n$.

5 Answer the following two questions : **14**

(1) Prove that, m is a σ-algebra on $\mathbb{R}$, where m is the family of all measurable sets on $\mathbb{R}$.

(2) Let $[a, b] \subseteq \mathbb{R}$.. Then prove that, $m^*([a, b]) = b - a$.

6 Answer the following two questions : **14**

(1) State and prove, Monotone Convergence Theorem.

(2) (a) Let D be a measurable subset of $\mathbb{R}$ and $E \subseteq D$. Let X_E is the characteristic function of D. Then prove that, X_E is a measurable function if and only if E is a measurable subset of $\mathbb{R}$.

(b) Let $f, g : E \to \mathbb{R}$ be two real valued functions on a measurable set E. Let f be a measurable function on E and $f = g$ a.e. on E. Then prove that, g is also a measurable function on E.

7 Answer the following two questions : **14**

(1) Let f, g be the non-negative Lebesgue integrable functions on a measurable set E. Let $c > 0$. Then prove that,

(i) $\int_E (cf) = c \int_E f$

(ii) $\int_E (f + g) = \int_E f + \int_E g.$

(2) State and prove, Fatou's Lemma.

8 Answer the following two questions : **14**

(1) Let $f : [0, 2] \to R$ be defined by $f(0) = 0$ and $f(x) = x \sin\left(\frac{\pi}{x}\right)$ for $x \neq 0$. Then show that, f is not a function of bounded variation on [0, 2].

(2) State and prove, Minkowski's Inequality.

9 Answer the following one question : **14**

Construct a non - measurable subset of [0,1] with required justification.

10 Answer the following one question : **14**

Let f, g be bounded measurable functions on E and $mE < \infty$. Then prove that,

(i) $\int_E (af + bg) = a\int_E f + b\int_E g, \ \forall a,\ b \in \mathbb{R}$

(ii) $f \le g$ a.e. on E then $\int_E f \le \int_E g$

(iii) $f = g$ a.e. on E then $\int_E f = \int_E g$

(iv) If $a \le f(x) \le b,\ \forall x \in E$, then $a \le \frac{1}{mE}\int_E f \le b$

(v) For any disjoint subset A and B of E, $\int_{A \cup B} f = \int_A f + \int_B f$.